International Journal of Theoretical Physics, Vol. 31, No. 8, 1992

$O(N)$ Symmetric Finite-Temperature φ^4 Theory in $2+1$ Dimensions

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Received November 25, 1991

Starting from the grand partition function, the symmetry breaking of a BE gas in $2+1$ dimension is investigated both analytically and numerically. For this the Gaussian effective potential method is extended to μ (chemical potential)-dependent $O(N)$ finite-temperature symmetric φ^4 theory. The relevant integrals involved are evaluated in a simple and straightforward manner. The results are compared with those of others obtained by the loop expansion method.

1. INTRODUCTION

In recent years quantum field theory in dimensions other than $3+1$ has become a focus of widespread research interest not only for academic and mathematical reasons, but because it is conjectured that these theories are capable of experimental predictions. In particular, three-dimensional dynamics is relevant for condensed matter (Belvedre, 1990; Dorey and Mavromatos, 1991; Schonfeld, 1981; Jackiw and Templeton, 1981). Campbell and Bishop (1982) discussed in detail the application of φ^4 theory in two space-time dimensions [in the case of dimerized polyacetylene $(CH)_x$]. Also, in $2+1$ dimensions the problem of precariousness can be avoided.

In this paper we make a Gaussian effective potential analysis (Stevenson, 1984, 1985, 1987; Stevenson *et al.*, 1986; Alles and Tarrach, 1986*a,b*; Stevenson and Roditi, 1986; Roditi, 1986*a,b*; Tarrach, 1986; Roy *et al.*, 1986*a,b*; Haber and Weldon, 1981, 1982) for a finite-temperature, μ-dependent, $O(N)$ symmetric $\lambda\varphi^4$ theory and investigate the symmetry breaking of a BE gas in $2+1$ dimensions. It is argued that the GEP (Stevenson, 1984, 1985, 1987; Stevenson *et al.*, 1986; Alles and Tarrach,

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0020-7748/92/0800-1389$06.50/0

1986*a,b*) essentially a nonperturbative approach, has several advantages over the loop expansion method. In the context of $O(N)$ symmetry it has been shown that the GEP reproduces the $1/N$ expansion result.

We have restricted ourselves to μ-dependent φ^4 theory in 2+1 dimensions, though our results can be applied to 1+1 dimensions after redefining the I_1 and I_0 integrals, The relevant integrals have been evaluated in a simple and straightforward manner. The method adopted here is quite different from that of Haber and Weldon (1981, 1982) based on a method suggested by Dolan and Jackiw (1974) and to the best of our knowledge has not been presented before.

It is well known that in 1+1 dimensions no phase transition occurs and it is argued by some authors (Dolan and Jackiw, 1974, and references therein) that in 2+1 dimensions the BE phase transition occurs for massless bosons but does not occur for massive bosons. These arguments were based on the fact that at low temperature the effective potential becomes minimum at $\varphi \neq 0$ only if the effective mass is equal to one of the chemical potentials. But in 1+1 and 2+1 dimensions the effective potential contains terms like $\mu \ln(M^2-\mu^2)$ (M is the effective mass of the system and μ is the chemical potential) and hence the only solution for symmetry breaking is given by $M=\mu=0$. But in the GEP approach the minimum at low temperature occurs when the variational mass satisfies a self-consistent equation and it is not obvious from the equation that symmetry breaking will not occur for massive bosons. One has to find a numerical solution to investigate the symmetry breaking.

This we have done explicitly in 2+1 dimensions, our starting point being the grand partition function Z. The GEP is essentially a nonperturbative variational method. In the present problem we have two variational parameters Ω and w, which make V_G a transcendental function of φ_0. Note that V_G gets contributions from divergent integrals such as $I_1(\Omega)$ and $I_0(\Omega)$. However, in 2+1 dimensions, as far as spontaneous symmetry breaking (SSB) is concerned, we did not get any nontrivial phase transition. However, even in 3+1 dimensions, where SSB occurs (Kapusta, 1981; Benson and Bernstein, 1991; Haber and Weldon, 1981, 1982), whether the critical temperature T_c will be qualitatively different in GEP theory from that obtained by others remains to be seen. As expected, our result is not different from that of Stevenson *et al.* (1987), as has been verified for $\mu=T=0$ [Stevenson *et al.*, (1987) did not consider the case $\mu \neq 0$]. But in 3+1 dimensions the GEP faces the precariousness problem (Stevenson, 1984, 1985, 1987; Stevenson *et al.*, 1986; Alles and Tarrach, 1986*a,b*; Stevenson and Roditi, 1986; Tarrach, 1986). To avoid this, Stevenson *et al.* (1987; Stevenson and Tarrach, 1986; Majumdar and Roychoudhury, 1992) have proposed an autonomous theory.

The plan of the paper is as follows.

In Section 2 the finite-temperature GEP including chemical potential for the $O(N)$ symmetric $\lambda\varphi^4$ theory is obtained for $2+1$ dimensions. In Section 3 we discuss the behavior of M (effective mass of the system) and μ with temperature. The phase transition aspects are analyzed on the basis of the expressions obtained and computational results.

In Section 4 calculations of $\bar{V}_G$ (finite-temperature GEP) are presented along with the behavior of $\bar{V}_G$ with φ_0 for different temperatures. Also in this section we write the expressions for pressure and thermodynamic potential in terms of $\bar{V}_G$. Finally, Section 5 includes discussions and remarks.

2. TEMPERATURE- AND μ-DEPENDENT GEP FOR $O(N)$ SYMMETRIC φ^4 THEORY

Before writing the Lagrangian for an interacting Bose gas with $O(N)$ non-Abelian symmetry, it should be mentioned that the chemical potential can be introduced only to mutually commutating generators rather than to each group generator (Turko, 1981). So for even N, the maximum number of mutually commutating charges is $N/2$. But in the present case we introduce a single chemical potential. Thus, our system is invariant only under $O(2)\times O(N-2)$, not under full symmetry. However, the generalization to more than one μ is straightforward and will be briefly mentioned in Section 5. The $O(N)$ symmetric Lagrangian for single μ is given by

$$\begin{aligned}\mathscr{L} &= \tfrac{1}{2}\partial_\nu\varphi_j\partial^\nu\varphi_j - \tfrac{1}{2}m_{\mathrm{B}}^2\varphi_j\varphi_j - \lambda_{\mathrm{B}}(\varphi_j\varphi_j)^2 \\ &\quad - i\mu(\dot{\varphi}_1\varphi_2 - \varphi_2\dot{\varphi}_1) + \tfrac{1}{2}\mu^2(\varphi_1^2+\varphi_2^2) \qquad (2.1)\end{aligned}$$

In the above and the folllowing equation, j is summed from 1 to N. The corresponding Hamiltonian is given by

$$\begin{aligned}H &= \tfrac{1}{2}\sum_j \dot{\varphi}_j^2 + \sum_j \tfrac{1}{2}(\nabla\varphi_j)^2 + \tfrac{1}{2}\sum_j m_{\mathrm{B}}^2\varphi_j^2 + \lambda_{\mathrm{B}}\sum_j (\varphi_j^2)^2 \\ &\quad + i\mu(\dot{\varphi}_1\varphi_2 - \varphi_1\dot{\varphi}_2) - \tfrac{1}{2}\mu^2(\varphi_1^2+\varphi_2^2) \qquad (2.2)\end{aligned}$$

For $O(N)$ symmetric theory, where only $(\varphi_0)_j$ sets a direction, the vibrational solution for the angles $\theta_1,\ldots,\theta_N$ will be such that the eigendirections of the Gaussian wave functional are radial and transverse and because of the remaining $O(N-1)$ symmetry, the $N-1$ transverse quantum fields would have equal mass parameter, say w, and the radial field would have a different mass parameters, say Ω (Stevenson *et al.*, 1987). To handle the present problem, we choose a coordinate system in which $(\varphi_0)_j$ points in the $j=1$ direction; then [writing $\varphi_j=(\varphi_0)_j+\hat{\varphi}_j$ and taking $(\varphi_0)_1=\varphi_0$]

$$\varphi^2 = \varphi_0^2 + 2\varphi_0\hat{\varphi}_1 + \sum \hat{\varphi}_j^2 \qquad (2.3)$$

where

$$\varphi_j = (\varphi_0)_j + R^j_i(\theta_1, \ldots, \theta_{N-1})\hat{\varphi}_j(\Omega_j), \qquad \varphi_0 = |\varphi_0|$$

and

$$(\varphi^2)^2 = \varphi_0^4 + 4\varphi_0^2\hat{\varphi}_1^2 + 2\varphi_0^2 \sum_1^N \varphi_j^2 + \left|\sum_1^N \hat{\varphi}_j^2\right|^2$$
$$+ 4\varphi_0^3\hat{\varphi}_1 + 4\varphi_0\hat{\varphi}_1 \sum_1^N \hat{\varphi}_j^2 \tag{2.4}$$

In the present case, due to existence of a net charge, the system will not be invariant under the full symmetry; rather, is invariant under $O(2)\times O(N-2)$. In this case, in general $(\varphi_0)_1$ and $(\varphi_0)_2$ are not equal to zero and we choose $(\varphi_0)_i = 0$ for $i = 3, 4, \ldots, N$. However, for the sake of simplicity we take $(\varphi_0)_2 = 0$, i.e., we take $(\varphi_0)_j = 0$ for $j = 2, \ldots, N$.

To compute ${}_{\Omega,w}\langle 0|H|0\rangle_{\Omega,w}$ and the finite-temperature GEP (FTGEP) $\bar{V}_G^{\mathrm{FT}}$ we start from basic premises and adopt a quasiperturbative approach (Hajj and Stevenson, 1988) to evaluate the partition function for finite temperature and chemical potential.

The Hamiltonian (2.2) is written as

$$H = H_0 + H_{\mathrm{int}}$$

In the present case H_0 and H_{int} may be written as

$$H_0 = \tfrac{1}{2}[\dot{\varphi}^2 + (\nabla\varphi)^2 + \Omega^2 \sum \hat{\varphi}_a^2 + w^2 \sum \hat{\varphi}_{a'}^2$$
$$+ i\mu(\dot{\varphi}_1\varphi_2 - \varphi_1\dot{\varphi}_2) - \mu^2(\hat{\varphi}_1^2 + \hat{\varphi}_2^2)] \tag{2.5}$$

where $a = 1, 2$ and $a' = 3, \ldots, N$, and

$$H_{\mathrm{int}} = -\tfrac{1}{2}[\Omega^2\hat{\varphi}_a^2 + w^2\hat{\varphi}_{\alpha'}^2 + \mu^2\varphi_0^2] + \tfrac{1}{2}m_{\mathrm{B}}^2(\varphi_0 + \hat{\varphi})^2$$
$$+ \lambda_{\mathrm{B}}(\varphi_0 + \hat{\phi})^4 \tag{2.5a}$$

In wiriting H_{int}, the result that $\langle\hat{\varphi}_1\rangle = 0$ has been taken into account. Now the grand partition function Z is given by

$$Z = \mathrm{Tr}\, e^{-\beta H} = \sum_\alpha \langle\alpha|e^{-\beta H}|\alpha\rangle$$

Again from the standard thermodynamic definition of the Helmholtz free energy $F = -(1/\beta)\ln Z$ with $\beta = 1/KT$, one can write

$$V_G^{\mathrm{FT}}(\varphi_0, \Omega, w, \mu) = \frac{F}{V} = -\frac{1}{\beta V}\ln Z_0 + \langle H_{\mathrm{int}}\rangle_T \tag{2.6}$$

where $Z_0 = \mathrm{Tr}\, e^{-\beta H_0}$ and $\langle H_{\mathrm{int}}\rangle_T$ is the thermal average of H_{int}. Now, since the effective potential of the system corresponds to the function of φ_0

resulting from the minimization of the free energy F under the constraint $\langle\varphi\rangle=\varphi_0$, we have

$$\tilde{V}_G^{\mathrm{FT}}(\varphi_0)=\min_{\Omega,w} V_G^{\mathrm{FT}}(\varphi_0, w, \Omega, \mu) \tag{2.7}$$

Now

$$\begin{aligned} &H_0|n_1, n_2, \ldots, n_1', n_2', \ldots\rangle \\ &\quad = I_1 V+n_1 w_1+n_2 w_2+\cdots+(N-2)[I_1' V+n_1' w_1'+n_2 w_2'+\cdots] \end{aligned} \tag{2.8}$$

where $|n_1, n_2, \ldots, n_1', n_2', \ldots\rangle$ represents the eigenstates of H_0 corresponding to n_i and n_i' quanta in the ith mode and the contributions from each mode give rise to vacuum energies $I_1 V$ and $(N-2)I_1' V$ with

$$I_1 V=V\int\frac{d^\nu k}{(2\pi)^\nu}\frac{w_k}{2}, \qquad I_1' V=V\int\frac{d^\nu k}{(2\pi)^\nu}\frac{w_k'}{2} \tag{2.9}$$

In deriving the above equations $\sum_i$ has been replaced by $V\int d^\nu k/(2\pi)^\nu$. Now, as a consequence of introduction of the chemical potential, the frequency w_i of the ith mode will have two values $w_i\to w_k=(k^2+\Omega^2)^{1/2}\pm\mu$ [as can be verified by solving the coupled Klein-Gordon equation involving fields associated with the chemical potential arising from the Hamiltonian (2.2)] instead of a single value $(k^2+\Omega^2)^{1/2}$ as in the case of the absence of μ. But since no chemical potential is associated with the remaining $N-2$ transverse fields, the corresponding frequency w_i' for the ith mode will be as usual single-valued but with a different mass parameter w. Thus, for μ-less fields $w_i'\to w_k'=(k^2+w^2)^{1/2}$.

The appearance of the factor $N-2$ in the second term on the rhs of equation (2.8) is the result of $N-2$ transverse quantum fields with the same mass parameter w.

Writing explicitly the trace appearing in Z_0, we have

$$Z_0=\sum_{\substack{n_1=0\text{ to }\infty\\ n_2=0\text{ to }\infty\\ \cdots}}\ \sum_{\substack{n_1'=0\text{ to }\infty\\ n_2'=0\text{ to }\infty\\ \cdots}}\langle n_1, n_2, \ldots, n_1', n_2', \ldots, |e^{-\beta H_0}|n, n'\rangle \tag{2.10}$$

where $|n, n'\rangle$ stands for $|n_1, n_2, \ldots, n_1', n_2', \ldots\rangle$ and the summation is extended over all modes i and for each mode the summation is over the occupation numbers n_i and n_i'. Therefore

$$\begin{aligned} Z_0&=e^{-\beta I_1 V}\left[\sum_{n_1=0}^{\infty} e^{-\beta n_1 w_1}\right]\left[\sum_{n_2=0}^{\infty} e^{-\beta n_2 w_2}\right]\cdots e^{-\beta I_1' V(N-2)} \\ &\quad\times\left[\sum_{n_1'=0}^{\infty} e^{-\beta(N-2)n_1' w_1'}\right]\left[\sum_{n_2'=0}^{\infty} e^{-\beta(N-2)n_2' w_2'}\right]\cdots \\ &=Z_0'(Z_0'')^{N-2} \end{aligned} \tag{2.11}$$

with

$$Z_0' = e^{-\beta I_1 V}\left[\sum_{n_1=0}^{\infty} e^{-\beta n_1 w_1}\right]\left[\sum_{n_2=0}^{\infty} e^{-\beta n_2 w_2}\right]\cdots$$
$$Z_0'' = e^{-\beta I_1' V}\left[\sum_{n_1'=0}^{\infty} e^{-\beta n_1' w_1'}\right]\left[\sum_{n_2'=0}^{\infty} e^{-\beta n_2' w_2'}\cdots\right] \tag{2.12}$$

Now Z_0' and Z_0'' may be written as

$$Z_0' = e^{-\beta I_1 V}\prod_i (1-e^{-\beta w_i})^{-1}$$
$$Z_0'' = e^{-\beta I_1' V}\prod_i (1-e^{-\beta w_i'})^{-1} \tag{2.13}$$

As already explained, w_k can be have two values $(k^2+\Omega^2)^{1/2}\pm\mu$. The two values of w_k can be attributed to the fact that $(k^2+\Omega^2)^{1/2}+\mu$ corresponds to a particle with charge +1 and $(k^2+\Omega^2)^{1/2}-\mu$ corresponds to a particle with charge −1. Again, since each of the w_k values is independent of the other, we have

$$Z_0' = e^{-\beta I_1 V}\prod_i (1-e^{-\beta (w_k)_1})^{-1}\prod_{i'} (1-e^{-\beta (w_k)_2})^{-1}$$

Therefore,

$$\ln Z_0' = -\beta I_1 V - \sum_i \ln(1-e^{-\beta (w_k)_1}) - \sum_{i'} \ln(1-e^{-\beta (w_k)_2}) \tag{2.14}$$

Defining

$$I_1^{\mathrm{FT}} = I_1(\Omega) + I_1^{\beta} = -\frac{1}{\beta V}\ln Z_0' \tag{2.15}$$

we have from equations (2.14) and (2.15)

$$I_1^{\beta} = \frac{1}{\beta}\int \frac{d^\nu k}{(2\pi)^\nu}[\ln(1-e^{-\beta(w_k)_1}) + \ln(1-e^{-\beta(w_k)_2})]$$
$$= \frac{1}{\beta}\int \frac{d^\nu k}{(2\pi)^\nu}[\ln(1-e^{-\beta(E+\mu)}) + \ln(1-e^{-\beta(E-\mu)})] \tag{2.16}$$

where $E=(k^2+\Omega^2)^{1/2}$; I_1^{β} is the sum of two zero-point energies of the charged particles.

Now $I_1(\Omega)$ is given by

$$I_1(\Omega) = \int \frac{d^\nu k}{(2\pi)^\nu}\frac{1}{2}[w_{k_1}+w_{k_2}]$$
$$= \int \frac{d^\nu k}{(2\pi)^\nu}(k^2+\Omega^2)^{1/2} \tag{2.17}$$

Now from (2.13), defining

$$I_1'^{\mathrm{FT}} = I_1'(w) + I_1'^{\beta} = -\frac{1}{\beta V}\ln Z_0$$

we get

$$\begin{aligned} I_1'^{\beta} &= \frac{1}{\beta}\int \frac{d^\nu k}{(2\pi)^\nu}\ln[1 - e^{-\beta(k^2+w^2)^{1/2}}] \\ I_1'(w) &= \int \frac{d^\nu k}{(2\pi)^\nu}\frac{1}{2}w_k' \end{aligned} \tag{2.18}$$

where $w_k' = (k^2 + w^2)^{1/2}$.

Starting from the definition of the thermal average

$$\langle \hat{\varphi}_\alpha^2\rangle_T = \frac{\sum_{\alpha_0}\langle \alpha_0|e^{-\beta H_0}\hat{\varphi}_\alpha^2|\alpha_0\rangle}{\sum_{\alpha_0}\langle \alpha_0|e^{-\beta H_0}|\alpha_0\rangle} \tag{2.19}$$

one obtains easily

$$\langle \hat{\varphi}_\alpha^2\rangle_T = I_0^{\mathrm{FT}} = I_0(\Omega) + I_0^{\beta} \tag{2.20}$$

with

$$\begin{aligned} I_0(\Omega) &= \int \frac{d^\nu k}{(2\pi)^\nu}\frac{1}{(k^2+\Omega^2)^{1/2}} \\ I_0^{\beta} &= 2\int \frac{d^\nu k}{(2\pi)^\nu}\frac{1}{(k^2+\Omega^2)^{1/2}}\left[\frac{1}{e^{\beta(E+\mu)}-1} + \frac{1}{e^{\beta(E-\mu)}-1}\right] \end{aligned} \tag{2.21}$$

Again I_1^{FT} and I_0^{FT} are related in the following way:

$$\frac{dI_1^{\mathrm{FT}}}{d\Omega} = \Omega I_0^{\mathrm{FT}} \tag{2.22}$$

Similarly, for transverse components we have $I_0'^{\mathrm{FT}} = I_0'(w) + I_0'^{\beta}$, the mass parameter being w. After obtaining contributions from each term of $\langle H_{\mathrm{int}}\rangle_T$ and using (2.3), (2.4), and (2.6) and also keeping in mind that our system is invariant under $O(2)\times O(N-2)$ symmetry (as already discussed), we get V_G. For convenience we henceforth write I_1^{FT} as I_1 and I_0^{FT} as I_0 and similarly $I_1'^{\mathrm{FT}}$ as I_1' and $I_0'^{\mathrm{FT}}$ as I_0'. Finally, we have

$$V_G = V_0 - \tfrac{1}{2}\mu^2\varphi_0^2 \tag{2.23}$$

with

$$\begin{aligned} V_0 = {} & [I_1 + \tfrac{1}{2}(m_{\mathrm{B}}^2 - \Omega^2)I_0] + (N-2)[I_1' + \tfrac{1}{2}(m_{\mathrm{B}}^2 - w^2)I_0'] \\ & + \tfrac{1}{2}m_{\mathrm{B}}^2\varphi_0^2 + \lambda_{\mathrm{B}}\varphi_0^4 + \lambda_{\mathrm{B}}[3I_0^2 + (N^2-2N)I_0'^2 \\ & + 2(N-2)I_0I_0' + 6I_0\varphi_0^2 + 2(N-2)I_0'\varphi_0^2] \end{aligned} \tag{2.24}$$

Now, minimization of $V_G(\varphi_0, \Omega, w, \mu)$ with respect to Ω and w and use of the results $dI_N/d\Omega = (2N-1)\Omega I_{N-1}$ and $dI'_N/dw = (2N-1)wI'_{N-1}$ enables us to write the following coupled equation for Ω and w:

$$\begin{aligned}\Omega^2 &= m_B^2 + 4\lambda_B[3I_0 + (N-2)I'_0 + 3\varphi_0^2]\\ w^2 &= m_B^2 + 4\lambda_B(I_0 + NI'_0 + \varphi_0^2)\end{aligned} \tag{2.25}$$

It can be easily shown that the GEP contains the leading-order term of $1/N$ expansions. Taking the limit $N \to \infty$ with $\lambda_B N$ and φ_0^2/N constant we get from equation (2.24)

$$\begin{aligned}\frac{V_G}{N} &= I'_1 + \frac{1}{2}(m_B^2 - w^2)I'_0 + \frac{1}{2}m_B^2\left(\frac{\varphi_0^2}{N}\right) + (N\lambda_B)\left(\frac{\varphi_0^2}{N}\right)^2\\ &\quad + (N\lambda_B)\left[I_0'^2 + 2I'_0\left(\frac{\varphi_0^2}{N}\right)\right] - \frac{1}{2}\mu^2\left(\frac{\varphi_0^2}{N}\right)\end{aligned} \tag{2.26}$$

In the above limit ($N \to \infty$) the w equation becomes

$$I'_0 = (w^2 - m_B^2)/4(\lambda_B N) - (\varphi_0^2/N) \tag{2.27}$$

Substituting (2.27) in (2.26), we get

$$\frac{V_G}{N} = I'_1 + \frac{1}{2}w^2\left(\frac{\varphi_0^2}{N}\right) - \frac{1}{16(N\lambda_B)}(w^2 - m_B^2)^2 - \frac{1}{2}\mu^2\left(\frac{\varphi_0^2}{N}\right) \tag{2.28}$$

The above result is nothing but the $1/N$ expansion result and putting $\mu = 0$, we get the case without μ already obtained (Stevenson *et al.*, 1987). Finally, the renormalized mass m_R, defined as the particle mass for $\varphi_0 = 0$, is given by

$$\begin{aligned}m_R^2 &= \left.\frac{d^2 V_0}{d\varphi_0^2}\right|_{\varphi_0=0} = 2\left.\frac{dV_0}{d\varphi_0^2}\right|_{\varphi_0=0}\\ &= m_B^2 + 4\lambda_B(N+1)I_0(m_R)\end{aligned} \tag{2.29}$$

3. CALCULATION OF MASS OF THE SYSTEM AND CHEMICAL POTENTIAL AND BEHAVIOR OF M AND μ WITH TEMPERATURE

The mass of the system is given by $d^2V_G/d\varphi_0^2$ evaluated at the minimum of the potential. Using equation (2.25) and the fact that $I_0^{\mathrm{FT}} = I_0(\Omega) + I_0^\beta$ and $I_0'^{\mathrm{FT}} = I'_0(w) + I_0'^\beta$, we get

$$\frac{d^2V_G}{d\varphi_0^2} = M^2 \text{ (say)} = \Omega^2 - \mu^2 \tag{3.1}$$

The coupled equation (2.25) now may be written in terms of the renormalized mass m_R^2,

$$\begin{aligned}\Omega^2 &= m_R^2 + 4\lambda_B\{3[I_0(\Omega) - I_0(m_R)] + (N-2)[I_0'(w) - I_0(m_R)] - 3\varphi_0^2\}\\ &\quad + 4\lambda_B[3I_0^\beta + (N-2)I_0'^\beta]\\ w^2 &= m_R^2 + 4\lambda_B\{I_0(\Omega) - I_0(m_R) + N[I_0'(w) - I_0(m_R)] - \varphi_0^2\}\\ &\quad + 4\lambda_B(I_0^\beta + NI_0'^\beta)\end{aligned} \tag{3.2}$$

Again we use the following results (Stevenson, 1984, 1985, 1987) for 2+1 dimensions:

$$\begin{aligned}I_1(\Omega) - I_1(m_R) &= -\frac{1}{2}(m_R^2 - \Omega^2)I_0(m_R) - m_R^3\frac{L_2(x)}{8\pi}\\ I_0(\Omega) - I_0(m_R) &= -\frac{m_R}{4\pi}L_1(x)\end{aligned} \tag{3.3}$$

with $L_1(x) = (\sqrt{x} - 1)$ and $L_2(x) = \frac{1}{3}(\sqrt{x} - 1)^2(2\sqrt{x} + 1)$, where $x = \Omega^2/m_R^2$ [a similar set of equations for $I_1'(w) - I_1(m_R)$ and $I_0'(w) - I_0(m_R)$ is obtained by replacing x by y, where $y = w^2/m_R^2$]. We get

$$\begin{aligned}\Omega^2 &= m_R^2 - 4\lambda_B\left[3m_R\frac{L_1(x)}{4\pi} + (N-2)m_R\frac{L_1(y)}{4\pi} - 3\varphi_0^2\right]\\ &\quad + 4\lambda_B[3I_0^\beta + (N-2)I_0'^\beta]\\ w^2 &= m_R^2 - 4\lambda_B\left[m_R\frac{L_1(x)}{4\pi} + Nm_R\frac{L_1(y)}{4\pi} - \varphi_0^2\right]\\ &\quad + 4\lambda_B(I_0^\beta + NI_0'^\beta)\end{aligned} \tag{3.4}$$

The integrals I_0^β and $I_0'^\beta$ appearing in the coupled equations (3.4) depend on (μ, Ω) and (μ, w), respectively. Thus, Ω, w, and μ are interdependent. From the definition of charge density and using (2.23)–(2.25), we have

$$\frac{dV_G}{d\mu} = \frac{\partial V_0}{\partial\Omega^2}\frac{d\Omega^2}{d\mu} - \frac{1}{2}\frac{d}{d\mu}(\mu^2\varphi_0^2)$$

So,

$$\rho = -\frac{dV_G}{d\mu} = -\left(\frac{\partial I_1^\beta}{\partial\mu}\right)_{(T,\Omega)\,\text{fixed}} + \mu\varphi_0^2 \tag{3.5}$$

The mass of the system can now be obtained by solving simultaneously the coupled equations (3.4) and (3.5) for fixed charge density and using equation (3.1) and the constraints imposed by the minimization of V_G. The condition of minimum V_G may be written with the help of equations (2.23) and (2.24) and we get

$$\frac{dV_G}{d\varphi_0}=0=\varphi_0[m_{\rm B}^2+4\lambda_{\rm B}\{\varphi_0^2+3I_0+(N-2)I_0'\}]-\mu^2 \tag{3.6}$$

Equation (3.6) suggests that for V_G to be minimum either $\varphi_0=0$ or

$$m_{\rm B}^2+4\lambda_{\rm B}[\varphi_0^2+3I_0+(N-2)I_0']=0 \tag{3.7}$$

The condition of minimum of V_G for $\varphi_0\neq 0$ may be written using (2.29) and (3.3) as follows:

$$\varphi_0^2=(\mu^2-m_R^2)+\frac{m_R}{4\pi}[3L_1(x)+(N-2)L_1(y)] \tag{3.8}$$

Thus, to obtain M^2 [equation (3.1)] one has to solve equations (3.4) and (3.5) simultaneously after substituting the value of φ_0^2 from equation (3.8) in the above equations. On the other hand, if a minimum occurs at $\varphi_0=0$, M^2 can be obtained by solving (3.4) and (3.5) after putting $\varphi_0=0$ in both the equations.

Substituting the value of $\partial I_1^\beta/\partial\mu$ from (A.13) of the Appendix, we get for $\varphi_0\neq 0$

$$\rho=\frac{\mu}{2\pi}\left[\frac{2}{\beta}-\frac{1}{\beta}\ln(\Omega^2-\mu^2)\beta^2+\frac{1}{36}(3\Omega^2-\mu^2)\beta\right]+\mu\varphi_0^2 \tag{3.9}$$

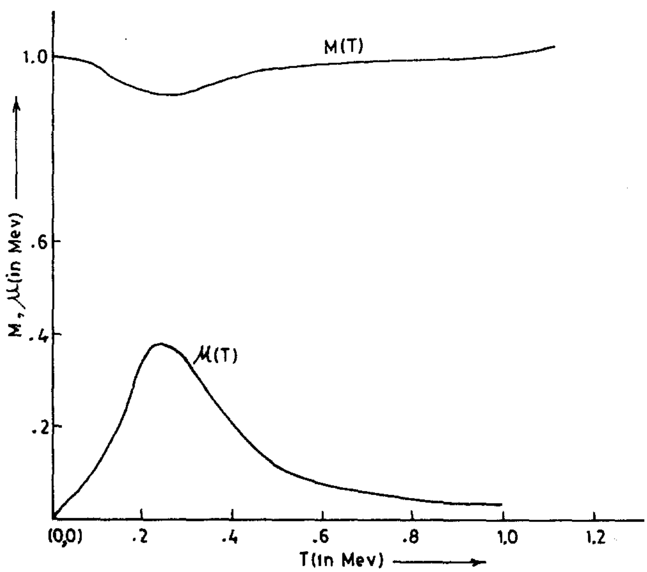

Fig. 1. Plots of $M(T)$ and $\mu(T)$ for $N=10$, $\rho=0.01$, and $M_R=1$ MeV.

Now the presence of the log term in equation (3.9) prevents Bose condensation for massive particle with $\Omega^2 \leq \mu^2$. However, whether any solution exists for $\Omega^2 > 0$ can only be checked by numerical analysis. As it happens, we do not find any nontrivial solution for $\varphi_0 \neq 0$ and for $\varphi_0 = 0$. We find solutions for M and μ for all temperatures without any break.

In Figure 1, M and μ are plotted against temperature. Since in $2+1$ dimensions I_{-1} is finite, normalization of λ does not present any problem and since λ is finite, for the sake of simplicity the $M(T)$ and $\mu(T)$ curves are drawn keeping λ fixed (we have taken here $\lambda = 0.01$). Also, m_R and ρ have been fixed at 1 and 0.01, respectively.

4. CALCULATION OF FINITE V_G AND ITS BEHAVIOR WITH φ_0 FOR DIFFERENT TEMPERATURES

A straightforward though lengthy calculation leads to a finite-temperature GEP for $2+1$ dimensions. Starting from equation (2.23) and then using (2.24), (2.25), (2.29), and (3.3), we get

$$
\begin{aligned}
\bar{V}_G = {} & -\frac{m_R^{\nu+1}}{8\pi} L_2(x) - \frac{m_R^{\nu+1}}{8\pi} L_1(x)[m_R^2(1-x)] \\
& -(N-2)\left\{\frac{m_R^{\nu+1}}{8\pi} L_2(y) - \frac{m_R^{\nu+1}}{8\pi} L_1(y)[m_R^2(1-y)]\right\} \\
& +\frac{m_R^2}{2}\varphi_0^2 + \lambda_B\varphi_0^4 + \lambda_B\left[\frac{3m_R^{2(\nu-1)}}{16\pi^2} L_1^2(x)\right. \\
& +(N^2-2N)\frac{m_R^{2(\nu-1)}}{16\pi^2} L_1^2(y) + 2(N-2)\frac{m_R^{2(\nu-1)}}{16\pi^2} L_1(x)L_1(y) \\
& \left. -\frac{6m_R^{\nu-1}}{4\pi} L_1(x)\varphi_0^2 - 2(N-2)\frac{m_R^{\nu-1}}{4\pi} L_1(y)\varphi_0^2\right] \\
& -\frac{\mu^2}{2}\varphi_0^2 + I_1^\beta + (N-2)I_1'^\beta \\
& -\lambda_B[3(I_0^\beta)^2 + (N^2-2N)(I_0'^\beta)^2 + 2(N-2)I_0^\beta I_0'^\beta] + D
\end{aligned}
\tag{4.1}
$$

where $\nu = 2$ (ν represents the space dimension) and D is usual divergent vacuum-energy term and is given by

$$D = (N-1)I_1(m_R) - \lambda_B(N^2-1)I_0(m_R) \tag{4.2}$$

As can be seen, the vacuum-energy term does not contain any temperature-dependent term.

The finite GEP is then obtained by substituting the values of $L_2(x)$, $L_1(x)$, $L_2(y)$, and $L_1(y)$ from (3.3) in equation (4.1) and the result is

$$\begin{aligned}
\bar{V}_G - D = &-\frac{m_R^3}{8\pi}\left[\frac{1}{3}(\sqrt{x}-1)^2(2\sqrt{x}+1)+(\sqrt{x}-1)(1-x)\right.\\
&\left.+(N-2)\left(\frac{1}{3}(\sqrt{y}-1)^2(2\sqrt{y}+1)+(\sqrt{y}-1)(1-y)\right)\right]\\
&+\frac{m_R^2\varphi_0^2}{2}+\lambda_B\varphi_0^4+\lambda_B\frac{m_R^2}{16\pi^2}\\
&\times\left[3(\sqrt{x}-1)^2+(N^2-2N)(\sqrt{y}-1)^2+2(N-2)(\sqrt{x}-1)(\sqrt{y}-1)\right.\\
&\left.-\frac{24\pi}{m_R}(\sqrt{x}-1)\varphi_0^2-\frac{8\pi}{m_R}(N-2)(\sqrt{y}-1)\varphi_0^2\right]\\
&-\frac{\mu^2}{2}\varphi_0^2+I_1^\beta+(N-2)I_1'^\beta\\
&-\lambda_B[3(I_0^\beta)^2+(N^2-2N)(I_0'^\beta)^2+2(N-2)I_0^\beta I_0'^\beta]
\end{aligned}\tag{4.3}$$

Note that (4.1) holds equally for 1+1 dimensions and substitution of values of $L_1(x)$, $L_2(x)$, $L_1(y)$, and $L_2(y)$ in this dimension would give the 1+1-dimensional $\bar{V}_G$. In Figure 2 we plot $\bar{V}_G$ with φ_0 for different temperatures. The curves in this figure also confirm the conclusion drawn in Section 3 regarding the phase transition, i.e., a phase transition does not occur for massive bosons. In the computation of $\bar{V}_G$ we solved the coupled equations (3.4) and (3.9) for a fixed λ and used the results of I_1^β, $I_1'^\beta$, I_0^β, $I_0'^\beta$ from the Appendix. The inputs of our calculations are $m_R = 1$, $\lambda = 0.01$, and $\rho = 0.01$, expressed in MeV and for $N = 10$.

Following the definitions of the thermodynamic potential $\hat{\Omega}$ and pressure P for $\varphi_0 = 0$, we have

$$\frac{\hat{\Omega}(T,\mu,V)}{V}=\bar{V}_G \qquad \text{and} \qquad P=-\bar{V}_G \tag{4.4}$$

Now using the definition of entropy S, we get

$$S=-\frac{d\bar{V}_G}{dT}=-\frac{\partial\bar{V}_0}{\partial\Omega^2}\frac{d\Omega^2}{dT}+\frac{\partial I_1^\beta}{\partial T}+\frac{d}{dT}\left(\frac{\mu^2}{2}\varphi_0^2\right) \tag{4.5}$$

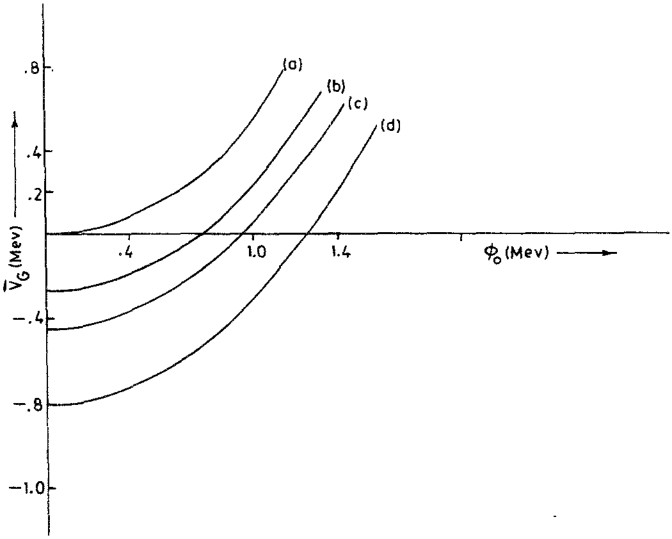

Fig. 2. Plot of $\bar{V}_G$ vs. φ_0 for $N=10$, $m_R=1$ MeV, $\rho=0.01$, $\lambda=0.01$, and (a) $T=0$, (b) $T=0.3$, (c) $T=0.6$, and (d) $T=0.8$ MeV.

In the above equation, detailed calculation shows, using (2.25) and (2.24), that the coefficient $d\Omega^2/dT$ vanishes. So, for $\varphi_0=0$ we get from (4.5) and (A.14)

$$S=\frac{1}{4\pi}\left[12\xi(3)T^2+2(2\mu^2-\Omega^2)+(\Omega^2-\mu^2)\ln(\Omega^2-\mu^2)/T^2 \right.$$
$$\left. +O(\Omega^4,\mu^4,\mu^2\Omega^2)\frac{1}{T^2}\right] \tag{4.6}$$

5. DISCUSSION AND CONCLUSIONS

In this paper we have considered a single μ. Our numerical results show that for finite ρ, a phase transition does not occur for nontrivial values of M and μ, as also observed by others (Haber and Weldon, 1981, 1982, and references therein) and there are two variational parameters Ω, w.

However, for $\mu=T=0$, our results are in agreement with that of Stevenson *et al.* (1987).

The extension to the many-μ case is straightforward. If we introduce all possible charges, one to the radial field and the remaining $(N/2-1)$ to the transverse fields, the Lagrangian and the corresponding Hamiltonian is modified as follows:

$$\mathscr{L} = \tfrac{1}{2}\partial_\gamma\varphi_j\partial^\gamma\varphi_j - \tfrac{1}{2}m_B^2\varphi_j\varphi_j - \lambda_B(\varphi_j\varphi_j)^2 - i\mu_k(\dot\varphi_{2k-1}\varphi_{2k} - \dot\varphi_{2k}\varphi_{2k-1}) + \tfrac{1}{2}\mu_k^2(\varphi_{2k-1}^2 + \varphi_{2k}^2) \tag{5.1}$$

$$H = \tfrac{1}{2}\sum_j \varphi_j^2 + \sum_j \tfrac{1}{2}(\nabla\varphi_j)^2 + \tfrac{1}{2}\sum_j m_B^2\varphi_j^2 + \lambda_B\sum_j(\varphi_j^2)^2 + i\sum_k \mu_k(\dot\varphi_{2k-1}\varphi_{2k} - \dot\varphi_{2k}\varphi_{2k-1}) - \tfrac{1}{2}\sum_k \mu_k^2(\varphi_{2k-1}^2 + \varphi_{2k}^2) \tag{5.2}$$

In the above equation, the sum over j is from 1 to N, and that for k is from 1 to $N/2$. The expression for V_G is

$$V_G = [I_1 + \tfrac{1}{2}(m_B^2 - \Omega^2)I_0] + \tfrac{1}{2}(N-2)[I_1' + \tfrac{1}{2}(m_B^2 - w^2)I_0'] + \tfrac{1}{2}m_B^2\varphi_0^2 + \lambda_B\varphi_0^4 + \lambda_B[3I_0^2 + \tfrac{1}{4}(N^2-4)I_0'^2 + (N-2)I_0I_0' + 6I_0\varphi_0^2 + (N-2)I_0'\varphi_0^2] - \tfrac{1}{2}\sum \mu_k^2\varphi_0^2 \tag{5.3}$$

and the corresponding equation for Ω^2, w^2 is obtained from equation (2.25) by dividing the I_0' terms by 2, and the integrals I_1' and I_0' are obtained just by replacing Ω by w in the I_1 and I_0 integrals. Further, the theory can be extended to gauge bosons and fermions.

APPENDIX. EVALUATION OF $I_1'^\beta$, $I_1'^\beta$, I_0^β, and $I_0'^\beta$

$$I_1^\beta(\Omega, \mu, T) = \frac{1}{2\pi\beta}\left\{\int_0^\infty k\,dk[\ln(1-e^{-\beta(E-\mu)}) + \ln(1-e^{-\beta(E+\mu)})\right\} \tag{A.1}$$

$$= \frac{1}{2\pi\beta}\int_\Omega^\infty E\,dE[\ln(1-e^{-\beta(E-\mu)}) + \ln(1-e^{-\beta(E+\mu)})]$$

$$= -\frac{1}{2\pi\beta}\sum_{n=1}^\infty\left[\int_\Omega^\infty \frac{e^{-n(E+\mu)\beta}}{n}E\,dE + \int_\Omega^\infty \frac{e^{-n(E-\mu)\beta}}{n}E\,dE\right]$$

$$= -\frac{1}{2\pi\beta}[I_1 + I_2] \tag{A.2}$$

with

$$I_1 = \sum_{n=1}^\infty \int_\Omega^\infty \frac{e^{-n(E-\mu)\beta}}{n}E\,dE \qquad I_2 = \sum_{n=1}^\infty \int_\Omega^\infty \frac{e^{-n(E+\mu)\beta}}{n}E\,dE \tag{A.3}$$

Now integrating (A.3), we get

$$I_1 = \Omega \sum_{n=1}^{\infty} \frac{e^{-n\beta(\Omega-\mu)}}{n^2\beta} + \sum_{n=1}^{\infty} \frac{e^{-n\beta(\Omega-\mu)}}{n^3\beta^2} \tag{A.4a}$$

Similarly,

$$I_2 = \Omega \sum_{n=1}^{\infty} \frac{e^{-n\beta(\Omega+\mu)}}{n^2\beta} + \sum_{n=1}^{\infty} \frac{e^{-n\beta(\Omega+\mu)}}{n^3\beta^2} \tag{A.4b}$$

Therefore,

$$\begin{aligned} I_1^\beta(\Omega, \mu, T) &= -\frac{1}{2\pi}\left[\frac{\Omega}{\beta^2}\sum_{n=1}^{\infty}\left(\frac{e^{-n(\Omega-\mu)\beta}}{n^2} + \frac{e^{-n(\Omega+\mu)\beta}}{n^2}\right)\right. \\ &\quad \left. + \frac{1}{\beta^3}\sum_{n=1}^{\infty}\left(\frac{e^{-n(\Omega-\mu)\beta}}{n^3} + \frac{e^{-n(\Omega+\mu)\beta}}{n^3}\right)\right] \\ &= -\frac{1}{2\pi}\left(\frac{\Omega}{\beta^2} J_1 + \frac{J_2}{\beta^3}\right) \end{aligned} \tag{A.5}$$

where

$$J_1 = \sum_{n=1}^{\infty}\left(\frac{e^{-n(\Omega-\mu)\beta}}{n^2} + \frac{e^{-n(\Omega+\mu)\beta}}{n^2}\right)$$

$$\frac{\partial J_1}{\partial \beta} = -\sum\left[(\Omega-\mu)\frac{e^{-n(\Omega-\mu)\beta}}{n} + (\Omega+\mu)\frac{e^{-n(\Omega+\mu)\beta}}{n}\right]$$

$$\begin{aligned} \frac{\partial J_1}{\partial \beta} &= (\Omega-\mu)\ln(1-e^{-\beta(\Omega-\mu)}) + (\Omega+\mu)\ln(1-e^{-\beta(\Omega+\mu)}) \\ &= (\Omega-\mu)\ln\left[\beta(\Omega-\mu) - \frac{\beta^2(\Omega-\mu)^2}{2} + \frac{\beta^2(\Omega-\mu)^3}{6}\cdots\right] \\ &\quad + (\Omega+\mu)\ln\left[\beta(\Omega+\mu) - \frac{\beta^2(\Omega+\mu)^2}{2} + \cdots\right] \end{aligned}$$

$$\begin{aligned} \frac{\partial J_1}{\partial \beta} &= (\Omega-\mu)\ln(\Omega-\mu)\beta + (\Omega-\mu)\ln(1+X) \\ &\quad + (\Omega+\mu)\ln(\Omega+\mu)\beta + (\Omega+\mu)\ln(1+X') \end{aligned} \tag{A.6}$$

with

$$X = -\frac{\beta(\Omega-\mu)}{2} + \frac{\beta^2(\Omega-\mu)^2}{6} - \cdots$$

and

$$X' = -\frac{\beta(\Omega+\mu)}{2} + \frac{\beta^2(\Omega+\mu)^2}{6} - \cdots$$

Now expanding $\ln(1+X)$ and $\ln(1+X')$ for small $|X|$ and $|X'|$ and integrating (A.6), we have, noticing that $J_1(\beta=0)=2\zeta(2)$,

$$J_1=\left[\beta(\Omega-\mu)\ln(\Omega-\mu)\beta-(\Omega-\mu)\beta-\frac{(\Omega-\mu)^2\beta^2}{4}+\frac{(\Omega+\mu)^3\beta^3}{72}-\cdots\right]+\left[\beta(\Omega+\mu)\ln(\Omega+\mu)\beta-\frac{(\Omega+\mu)^2\beta^2}{4}+\frac{(\Omega+\mu)^3\beta^3}{72}-\cdots\right]+2\zeta(2) \tag{A.7}$$

where $\zeta(P)$ is the Riemann zeta function defined by

$$\zeta(P)=\sum_{n=1}^{\infty}\frac{1}{n^P}$$

Now,

$$J_2=\sum_{n=1}^{\infty}\left(\frac{e^{-n\beta(\Omega-\mu)}}{n^3}+\frac{e^{-n\beta(\Omega+\mu)}}{n^3}\right)$$

or

$$\frac{\partial J_2}{\partial\beta}=-\left[(\Omega-\mu)\sum_{n=1}^{\infty}\frac{e^{-n\beta(\Omega-\mu)}}{n^2}+(\Omega+\mu)\sum\frac{e^{-n\beta(\Omega+\mu)}}{n^2}\right]$$

Again proceeding in the above manner, we get

$$J_2=-\left\{(\Omega-\mu)^2\left[\frac{\beta^2}{2}\ln(\Omega-\mu)\beta-\frac{3\beta^2}{4}-\frac{\Omega-\mu}{12}\beta^3+\frac{(\Omega-\mu)^2\beta^4}{288}-\cdots\right]+(\Omega+\mu)^2\left[\frac{\beta^2}{2}\ln(\Omega+\mu)\beta-\frac{3\beta^2}{4}-\frac{(\Omega+\mu)\beta^3}{12}+\frac{(\Omega+\mu)^2\beta^4}{288}-\cdots\right]\right\}-2\Omega\beta\zeta(2)+2\zeta(3) \tag{A.8}$$

Now using (A.7) and (A.8), we get from (A.5)

$$I_1^\beta=-\frac{1}{2\pi}\left\{\frac{\Omega^2-\mu^2}{2\beta}\ln\beta^2(\Omega^2-\mu^2)+\frac{3\mu^2-\Omega^2}{2\beta}-\frac{\Omega^3}{3}+\left[\frac{\Omega}{36}(\Omega^3+3\mu^2)-\frac{1}{288}(\Omega+\mu)^4+(\Omega-\mu)^4\right]\beta+\frac{2\zeta(3)}{\beta^3}+O(\beta)^2+\cdots\right\} \tag{A.9}$$

From the above results, one can find easily $I_1'^{\beta}(w, T)$ (put $\mu=0$ and divide the result by 2),

$$I_1'(w) = -\frac{1}{2\pi}\left[\frac{w^2}{2\beta}\ln\beta w - \frac{w^2}{4\beta} - \frac{w^3}{6} + \frac{w^4}{96}\beta + \frac{\zeta(3)}{\beta^3} + O(\beta^2) + \cdots\right] \tag{A.10}$$

Finally, using (A.6) and (A.7), we get I_0^{β}, $I_0'^{\beta}$, $\partial I_1^{\beta}/\partial\mu$, and $\partial I_1^{\beta}/\partial T$, and the results are $(T=1/\beta)$

$$I_0^{\beta}(\Omega, \mu, T) = \frac{1}{\Omega}\frac{\partial I_1^{\beta}}{\partial\Omega} = -\frac{1}{2\pi}\left[\frac{1}{\beta}\ln\beta^2(\Omega^2-\mu^2) - \Omega + \frac{1}{12}(\Omega^2+\mu^2)\beta - \cdots\right] \tag{A.11}$$

Similarly,

$$I_0'^{\beta}(T, w) = -\frac{1}{2\pi}\left(\frac{\ln\beta w}{\beta} - \frac{w}{2} + \frac{w^2}{24}\beta + \cdots\right) \tag{A.12}$$

$$\frac{\partial I_1^{\beta}}{\partial\mu} = -\frac{\mu}{2\pi}\left[\frac{2}{\beta} - \frac{\ln\beta^2}{\beta}(\Omega^2-\mu^2) + \frac{1}{36}(3\Omega^2-\mu^2)\beta\cdots\right] \tag{A.13}$$

$$\frac{\partial I_1^{\beta}}{\partial T} = -\frac{1}{4\pi}[12\zeta(3)T^2 + 2(2\mu^2-\Omega^2) + (\Omega^2-\mu^2) \times\ln(\Omega^2-\mu^2)/T^2 + O(1/T^2) \tag{A.14}$$

ACKNOWLEDGMENT

B.K.M. is grateful to the Physics Department, B.K.C. College, Calcutta, for providing computational facilities.

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